

COMPACT EMBEDDED λ -TORUS IN EUCLIDEAN SPACES

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ABSTRACT. In this paper, we construct compact embedded λ -hypersurfaces with the topology of torus which are called λ -torus in Euclidean spaces \mathbb{R}^{n+1} .

1. INTRODUCTION

Let $X : M \rightarrow \mathbb{R}^{n+1}$ be an n -dimensional smooth immersed hypersurface in the $(n+1)$ -dimensional Euclidean space \mathbb{R}^{n+1} . In [4], Cheng and Wei have introduced the weighted volume-preserving mean curvature flow, which is defined as the following: a family $X(t)$ of smooth immersions:

$$X(t) = X(\cdot, t) : M \rightarrow \mathbb{R}^{n+1}$$

with $X(0) = X(\cdot, 0) = X(\cdot)$ is called a *weighted volume-preserving mean curvature flow* if they satisfy

$$(1.1) \quad \frac{\partial X(t)}{\partial t} = -\alpha(t)N(t) + \mathbf{H}(t),$$

where

$$\alpha(t) = \frac{\int_M H(t) \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu},$$

$\mathbf{H}(t) = \mathbf{H}(\cdot, t)$ and $N(t)$ denote the mean curvature vector and the unit normal vector of hypersurface $M_t = X(M^n, t)$ at point $X(\cdot, t)$, respectively and N is the unit normal vector of $X : M \rightarrow \mathbb{R}^{n+1}$.

One can prove that the flow (1.1) preserves the weighted volume $V(t)$ defined by

$$V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu.$$

The *weighted area functional* $A : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ is defined by

$$A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu_t,$$

where $d\mu_t$ is the area element of M in the metric induced by $X(t)$.

Let $X(t) : M \rightarrow \mathbb{R}^{n+1}$ with $X(0) = X$ be a variation of X . If $V(t)$ is constant for any t , we call $X(t) : M \rightarrow \mathbb{R}^{n+1}$ a *weighted volume-preserving variation of X* . Cheng and Wei [4] have proved that $X : M \rightarrow \mathbb{R}^{n+1}$ is a critical point of the weighted area

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functional $A(t)$ for all weighted volume-preserving variations if and only if there exists constant λ such that

$$(1.2) \quad \langle X, N \rangle + H = \lambda.$$

An immersed hypersurface $X(t) : M \rightarrow \mathbb{R}^{n+1}$ is called a λ -hypersurface of weighted volume-preserving mean curvature flow if the equation (1.2) is satisfied.

Remark 1.1. If $\lambda = 0$, λ -hypersurfaces are self-shrinkers of mean curvature flow. Hence, λ -hypersurfaces are a generalization of self-shrinkers of the mean curvature flow. For research on self-shrinkers of mean curvature flow, see [2, 3, 5, 6, 9].

Example 1.1. The n -dimensional sphere $S^n(r)$ with radius $r > 0$ is a compact λ -hypersurface in \mathbb{R}^{n+1} with $\lambda = \frac{n}{r} - r$.

Example 1.2. For $1 \leq k \leq n-1$, the n -dimensional cylinder $S^k(r) \times \mathbb{R}^{n-k}$ with radius $r > 0$ is a complete and non-compact λ -hypersurface in \mathbb{R}^{n+1} with $\lambda = \frac{k}{r} - r$.

Example 1.3. The n -dimensional Euclidean space \mathbb{R}^n is a complete and non-compact λ -hypersurface in \mathbb{R}^{n+1} with $\lambda = 0$.

It is well known that many interesting examples of compact self-shrinker of mean curvature flow are found recently. For instance, Angenent [1] has constructed compact embedded self-shrinker

$$X : S^1 \times S^{n-1} \rightarrow \mathbb{R}^{n+1}.$$

Drugan [8] has discovered self-shrinker, which is a topological sphere

$$X : S^2 \rightarrow \mathbb{R}^3$$

and complete self-shrinkers with higher genus in \mathbb{R}^3 are constructed by Kapouleas, Kleene and Møller [10] (see [11] also) and Nguyen [12]-[14].

Our purpose in this paper is to construct compact embedded λ -hypersurfaces in \mathbb{R}^{n+1} .

Theorem 1.1. For $n \geq 2$ and any $\lambda > 0$, there exists an embedded rotational λ -hypersurface

$$(1.3) \quad X : M \rightarrow \mathbb{R}^{n+1},$$

which is called λ -torus.

2. EQUATIONS OF ROTATIONAL λ -HYPERSURFACES IN \mathbb{R}^{n+1}

Let $\gamma(s) = (x(s), r(s))$, $s \in (a, b)$ be a curve with $r > 0$ in the upper half plane $\mathbb{H} = \{x + ir \mid r > 0, x \in \mathbb{R}, i = \sqrt{-1}\}$, where s is arc length parameter of $\gamma(s)$. We consider a rotational hypersurface $X : (a, b) \times S^{n-1}(1) \hookrightarrow \mathbb{R}^{n+1}$ in \mathbb{R}^{n+1} defined by

$$(2.1) \quad X : (a, b) \times S^{n-1}(1) \hookrightarrow \mathbb{R}^{n+1}, \quad X(s, \alpha) = (x(s), r(s)\alpha) \in \mathbb{R}^{n+1}$$

where $S^{n-1}(1)$ is the $(n-1)$ -dimensional unit sphere (cf. [7]).

By a direct calculation, one has the unit normal vector

$$(2.2) \quad N = (-r', x'\alpha)$$

and the mean curvature

$$(2.3) \quad H = -x''r' + x'r'' - \frac{n-1}{r}x'.$$

Therefore, we know from (2.1) and (2.2)

$$(2.4) \quad \langle X, N \rangle = -xr' + rx'.$$

Hence, $X : (a, b) \times S^{n-1}(1) \hookrightarrow \mathbb{R}^{n+1}$ is a λ -hypersurface in \mathbb{R}^{n+1} , if and only if, from (1.2), (2.3) and (2.4),

$$(2.5) \quad -x''r' + x'r'' - \frac{n-1}{r}x' - xr' + rx' = \lambda.$$

Since s is arc length parameter of the profile curve $\gamma(s) = (x(s), r(s))$, we have

$$(2.6) \quad (x')^2 + (r')^2 = 1,$$

Thus, it follows that

$$(2.7) \quad x'x'' + r'r'' = 0.$$

The signed curvature $\kappa(s)$ of the profile curve $\gamma(s) = (x(s), r(s))$ is given by

$$(2.8) \quad \kappa(s) = -\frac{x''}{r'}.$$

and it is known that the integral of the signed curvature $\kappa(s)$ measures the total rotation of the tangent vector of $\gamma(s)$. From (2.5) and (2.7), one has

$$(2.9) \quad x'' = -r' \left[xr' + \left(\frac{n-1}{r} - r \right) x' + \lambda \right].$$

Hence, we have

$$(2.10) \quad \begin{cases} (x')^2 + (r')^2 = 1 \\ -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r \right) x' + \lambda. \end{cases}$$

First of all, we consider several special solutions of (2.10).

- (1) $(x, r) = (0, s)$ is a solution.

This curve corresponds to the hyperplane through $(0, 0)$ and $\lambda = 0$.

- (2) $(x, r) = (a \cos \frac{s}{a}, a \sin \frac{s}{a})$ is a solution, where $a = \frac{\sqrt{\lambda^2 + 4n} - \lambda}{2}$.

This circle $x^2 + r^2 = a^2$ corresponds to a sphere $S^n(a)$ with radius a .

- (3) $(x, r) = (-s, a)$ is a solution, where $a = \frac{\sqrt{\lambda^2 + 4(n-1)} - \lambda}{2}$.

This straight line corresponds to a cylinder $S^{n-1}(a) \times \mathbb{R}$.

Next, we consider to find several other solutions of (2.10) besides the above three special solutions. In fact, our purposes are to study properties of the profile curve and to find a simple and closed profile curve γ in the upper half plane $\mathbb{H} = \{x+ir \mid r > 0, x \in \mathbb{R}, i = \sqrt{-1}\}$.

From now, we consider general behavior of profile curve γ . As long as $x > 0$, $x' > 0$ and $r < \frac{\sqrt{\lambda^2 + 4(n-1)} + \lambda}{2}$, one has from (2.8) and (2.10) that

$$\kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > 0,$$

that is, the curve γ bends upwards, so that the curve will be convex.

At $s = 0$, the profile curve $\gamma_\delta(s) = (x_\delta(s), r_\delta(s))$ with $(x_\delta(0), r_\delta(0)) = (0, \delta)$ and the initial unit tangent vector $(x'_\delta(0), r'_\delta(0)) = (1, 0)$ will initially bend upwards, where $\delta < \frac{\sqrt{\lambda^2 + 4(n-1)} + \lambda}{2}$. We give the following definition of s_1 .

Definition 2.1. Let $s_1 = s_1(\delta) > 0$ be the arc length of the first time, if any, at which either $x_\delta = 0$ or the unit tangent vector is $(1, 0)$, or $(-1, 0)$, that is, either the curve γ_δ hits r -axis, or the curve γ_δ has a horizontal tangent. If this never happens, we take $s_1(\delta) = S(\delta)$, where $\gamma_\delta = (x_\delta, r_\delta) : [0, S(\delta)) \rightarrow \mathbb{R}^2$ is the maximal solution of (2.10) with initial value $(x_\delta(0), r_\delta(0), x'_\delta(0)) = (0, \delta, 1)$.

From this definition 2.1, we have $r'(s) > 0$ in $(0, s_1)$. Hence, this curve can be written as a graph of $x = f_\delta(r)$, where $\delta < r < r_\delta(s_1)$. If $f'_\delta(r) = \frac{dx}{dr} = 0$, i.e., the profile curve γ_δ has a vertical tangent, then it follows from (2.10) that

$$\begin{aligned} f''_\delta(r) &= \frac{d^2 f_\delta(r)}{dr^2} = -\frac{1}{(r')^3} [xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda] \\ (2.11) \quad &= -\frac{1}{(r')^3} (xr' + \lambda) < 0. \end{aligned}$$

This means that $f_\delta(r)$ can only have local maximum. Thus, $f_\delta(r)$ can have at most one critical point, which must be a maximum point.

Next, we shall prove that there exist $\delta > 0$ such that $\gamma_\delta([0, s_1(\delta)])$ is a simple curve in the first quadrant which begins and ends on the r -axis, and whose tangent vectors on the r -axis are horizontal. From (2.10), one can get that the profile curve γ obtained by reflecting $\gamma_\delta([0, s_1(\delta)])$ in the r -axis is a simple and closed curve in the upper half plane.

3. AN ESTIMATE ON UPPER BOUNDS OF $r_\delta(s_1)$

We will consider behavior of profile curve γ as $\delta > 0$ is small enough in order to estimate supremum of $r_\delta(s_1)$. Since δ is very small, we define

$$(3.1) \quad \begin{cases} \xi(t) = \frac{1}{\delta}x(\delta t) \\ \rho(t) = \frac{1}{\delta}(r(\delta t) - \delta). \end{cases}$$

From (2.10), we have

$$(3.2) \quad \begin{cases} (\xi')^2 + (\rho')^2 = 1 \\ \frac{\xi''}{-\rho'} = \frac{\xi''}{-\sqrt{1 - (\xi')^2}} \\ = \delta^2 \xi \rho' + \left(\frac{n-1}{1+\rho} - \delta^2(1+\rho) \right) \xi' + \lambda \delta \\ = \frac{n-1}{1+\rho} \xi' + \lambda \delta + O(\delta^2) \end{cases}$$

and

$$(3.3) \quad \xi(0) = 0, \quad \rho(0) = 0, \quad \xi'(0) = 1.$$

We consider equations

$$(3.4) \quad \begin{cases} (\xi')^2 + (\rho')^2 = 1 \\ \frac{\xi''}{-\rho'} = \frac{\xi''}{-\sqrt{1 - (\xi')^2}} = \frac{n-1}{1+\rho} \xi', \end{cases}$$

with

$$(3.5) \quad \xi(0) = 0, \quad \rho(0) = 0, \quad \xi'(0) = 1.$$

From (3.4), one gets

$$1 - (\rho')^2 = \frac{1}{(1+\rho)^{2(n-1)}}.$$

If $\rho(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, we have

$$\rho' = \frac{d\rho}{dt} = \frac{dr}{ds} \rightarrow 1.$$

If $\rho(t)$ is bounded, we have

$$(3.6) \quad c_1 \leq \frac{\xi''}{-\xi' \sqrt{1 - (\xi')^2}} = \frac{n-1}{1+\rho} \leq (n-1)$$

where $c_1 > 0$ is a constant. By a direct calculation, we get

$$(3.7) \quad \tanh(c_1 t) \leq \sqrt{1 - (\xi')^2} = \rho'.$$

Hence, $\rho' \rightarrow 1$ as $t \rightarrow +\infty$.

Since the solution of (3.2) depends smoothly on the parameter δ , we may obtain from (3.7) that there is a $T > 0$ such that for all sufficiently small $\delta > 0$, one has $T\delta < S(\delta)$ and at $s = T\delta$,

$$(3.8) \quad r'_\delta(T\delta) \geq \begin{cases} \frac{\sqrt{3}}{2}, & \text{if } \lambda > \frac{\pi}{3\sqrt{n-1}} \\ \sin\left(\frac{\pi}{2} - \lambda \frac{\sqrt{n-1}}{2}\right), & \text{if } \lambda \leq \frac{\pi}{3\sqrt{n-1}}. \end{cases}$$

and $x_\delta = O(\delta)$, $r_\delta = \delta + O(\delta)$ from (3.1).

Lemma 3.1. *For $0 < s < s_1$, we have*

$$(3.9) \quad x_\delta(s) \leq C_1\delta, \quad r_\delta(s) \leq \sqrt{n-1} + \frac{\pi}{2\lambda} \text{ and } s_1(\delta) < \infty$$

if δ is small enough, where constant C_1 does not depend on δ .

Proof. For $0 < s < s_1 = s_1(\delta)$, we have $r' > 0$, so that γ_δ can be written as a graph of $x = f_\delta(r)$. From (2.10), we have

$$(3.10) \quad \begin{aligned} -\frac{x''}{r'} &= xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda \\ &\geq \left(\frac{n-1}{r} - r\right)x' \end{aligned}$$

because of $x > 0$, $\lambda > 0$ and $r' > 0$.

Letting $r_T = r_\delta(T\delta)$ and for r satisfying $x'(\tilde{r}) > 0$ if $r_T < \tilde{r} < r$, integrating (3.10) from r_T to $r > r_T$, we have

$$(3.11) \quad \frac{x'(s(r_T))}{x'(s(r))} \geq e^{\frac{r_T^2 - r^2}{2}} \left(\frac{r}{r_T}\right)^{n-1},$$

i.e., for all r such that $x' > 0$ on (r_T, r) , we have

$$(3.12) \quad x'(s(r)) \leq \left(\frac{r_T}{r}\right)^{n-1} e^{\frac{r^2 - r_T^2}{2}} x'(s(r_T)).$$

We will prove that the above r satisfies

Claim: $r \leq \sqrt{n-1}$ if δ is small enough.

If $r(s_1) \leq \sqrt{n-1}$, then we know that this result is obvious because of $r'(s) > 0$.

We only need to consider the case of $r(s_1) > \sqrt{n-1}$. In this case, there exists an s_3 such that $0 < s_3 < s_1$ and $r(s_3) = \sqrt{n-1}$.

If $x'(s_3) \leq 0$, we have $x'(s) < 0$ for $s_3 < s < s_1$. Hence, $r < r(s_3) = \sqrt{n-1}$ holds because of $x'(r) > 0$.

If $x'(s_3) > 0$, for $s \in (T\delta, s_3)$, $r'(s) > 0$ holds. One has $0 < r(s) < \sqrt{n-1}$ and

$$(3.13) \quad \kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda \geq \lambda$$

because of $r(s_3) = \sqrt{n-1}$. By integrating (3.13) from $s = T\delta$ to s_3 , we obtain

$$(3.14) \quad \int_{T\delta}^{s_3} \kappa ds \geq \lambda(s_3 - T\delta) \geq \lambda(r(s_3) - r(T\delta)) \geq \lambda(\sqrt{n-1} - r_T).$$

Here we use that the length of the profile curve $\gamma(s)$ from the point $\gamma(T\delta)$ to the point $\gamma(s_3)$ is not less than the Euclidean distance between these two points. Therefore, we have

$$(3.15) \quad r_T \geq \sqrt{n-1} - \frac{1}{\lambda} \int_{T\delta}^{s_3} \kappa ds.$$

If $\lambda > \frac{\pi}{3\sqrt{n-1}}$, we have

$$\frac{\pi}{2} - \lambda \frac{\sqrt{n-1}}{2} < \frac{\pi}{3}.$$

If $\lambda \leq \frac{\pi}{3\sqrt{n-1}}$, then we have

$$\frac{\pi}{2} - \lambda \frac{\sqrt{n-1}}{2} \geq \frac{\pi}{3}.$$

Since the integral

$$\int_{T\delta}^{s_3} \kappa ds$$

measures the total rotation of the tangent vector of γ_δ from $T\delta$ to s_3 , from (3.8), we know

$$\int_{T\delta}^{s_3} \kappa ds \leq \frac{\pi}{2} - \max\left\{\frac{\pi}{3}, \left(\frac{\pi}{2} - \lambda \frac{\sqrt{n-1}}{2}\right)\right\}.$$

Thus, for $\lambda > \frac{\pi}{3\sqrt{n-1}}$, we obtain from (3.15)

$$(3.16) \quad r_T \geq \sqrt{n-1} - \frac{1}{\lambda} \int_{T\delta}^{s_3} \kappa ds \geq \sqrt{n-1} - \frac{1}{\lambda} \left(\frac{\pi}{2} - \frac{\pi}{3}\right) \geq \frac{\sqrt{n-1}}{2}.$$

It is impossible because $r_T = r(T\delta) = O(\delta)$ is very small.

For $\lambda \leq \frac{\pi}{3\sqrt{n-1}}$, we have from (3.15)

$$(3.17) \quad r_T \geq \sqrt{n-1} - \frac{1}{\lambda} \int_{T\delta}^{s_3} \kappa ds \geq \sqrt{n-1} - \frac{1}{\lambda} \left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \lambda \frac{\sqrt{n-1}}{2}\right)\right) = \frac{\sqrt{n-1}}{2}.$$

It is also impossible because of $r_T = r(T\delta) = O(\delta)$.

From the above arguments, we complete the proof of the claim.

By a direct calculation, we have from (3.12) and (3.8)

$$(3.18) \quad x'(s(r)) \leq x'(s(\delta_T)) \leq \frac{1}{2}$$

for $r \in (r_T, \sqrt{n-1})$ with $x' > 0$ on (r_T, r) since

$$\left(\frac{r_T}{r}\right)^{n-1} e^{\frac{r^2 - r_T^2}{2}}$$

is a decreasing function of r in $(r_T, \sqrt{n-1})$. Hence, for $r \in (r_T, \sqrt{n-1})$ one has

$$(3.19) \quad \frac{x'}{r'} \leq 2x'.$$

Since $\frac{dx}{dr} = f'_\delta(r) = \frac{x'}{r'}$, we have from (3.12) and (3.19) that

$$(3.20) \quad \int_{r_T}^r f'_\delta(r) dr = \int_{r_T}^r \frac{x'}{r'} dr \leq 2 \int_{r_T}^r x' dr \leq 2 \int_{r_T}^r \left(\frac{r_T}{r}\right)^{n-1} e^{\frac{r^2 - r_T^2}{2}} x'(s(r_T)) dr.$$

Hence, we obtain

$$(3.21) \quad \begin{aligned} f_\delta(r) &\leq f_\delta(r_T) + 2 \int_{r_T}^r \left(\frac{r_T}{r}\right)^{n-1} e^{\frac{r^2 - r_T^2}{2}} dr \\ &\leq f_\delta(r_T) + c_2 \int_{r_T}^r \left(\frac{r_T}{r}\right)^{n-1} dr \leq c_3 r_T \leq C_1 \delta \end{aligned}$$

if $\delta > 0$ is small enough, where c_2 , c_3 and C_1 are constants.

Since $x_\delta(s)$ gets its maximum at $x'(s) = 0$, we conclude from (3.21) that

$$(3.22) \quad 0 \leq x_\delta(s) \leq C_1 \delta, \quad \text{for } 0 < s < s_1.$$

If $x'(s_3) = 0$ at $s = s_3$, then we have $x'(s) < 0$ for $s_3 < s < s_1$. According to the argument in the above claim, we know $r(s_3) \leq \sqrt{n-1}$ as long as $\delta > 0$ is small enough. If there exists $s_3 < s_2 < s_1$ such that $r(s_2) = \sqrt{n-1}$, then $r(s) > \sqrt{n-1}$ and $x'(s) < 0$ for $s_2 < s < s_1$. Hence, we have

$$\left(\frac{n-1}{r(s)} - r(s)\right)x'(s) > 0$$

and

$$(3.23) \quad \kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > \lambda,$$

for $s_2 < s < s_1$. By integrating (3.23) from $s = s_2$ to s_1 , we have

$$(3.24) \quad \frac{\pi}{2} \geq \int_{s_2}^{s_1} \kappa ds \geq \lambda(s_1 - s_2) \geq \lambda(r(s_1) - r(s_2)) = \lambda(r(s_1) - \sqrt{n-1}),$$

that is,

$$(3.25) \quad r_\delta(s_1) \leq \sqrt{n-1} + \frac{\pi}{2\lambda}.$$

Hence, we get that $x_\delta(s)$ and $r_\delta(s)$ are bounded. We know $s_1(\delta) < \infty$. This finishes the proof of the lemma 3.1. \square

Lemma 3.2. *For $\delta > 0$ small enough, one has $x_\delta(s_1) = 0$.*

Proof. If not, there exists a sequence $\delta_m \rightarrow 0^+$ for which $x_{\delta_m}(s) > 0$. Putting $f_m(r) = f_{\delta_m}(r)$, the function $x_{\delta_m} = f_m(r)$ is defined for $\delta_m < r < \sqrt{n-1} + \frac{\pi}{2\lambda}$ and satisfies

$$(3.26) \quad \frac{f_m''(r)}{1 + (f_m'(r))^2} + \left(\frac{n-1}{r} - r\right)f_m'(r) + f_m(r) + \frac{\lambda}{\sqrt{1 + (f_m'(r))^2}} = 0.$$

From the lemma 3.1, we get that $f_m(r)$ satisfies $0 < f_m(r) = x_{\delta_m}(r) \leq C_1 \delta_m \rightarrow 0$. Thus, we know that $\gamma_{\delta_m} = (x_{\delta_m}, r_{\delta_m})$ gets close to the r -axis, its tangents also must converge to the r -axis. Hence, we have $x_{\delta_m}(r) = f_m(r)$ and $f_m'(r)$ converge to zero on compact intervals.

On the other hand, from (3.26), we have

$$(3.27) \quad f_m''(r) \rightarrow -\lambda < 0.$$

Therefore, for $\delta_m > 0$ small enough,

$$(3.28) \quad f_m''(r) < -\frac{\lambda}{2} < 0,$$

$f_m'(r)$ is a monotone decreasing function. This is impossible. Hence, there exists $\delta > 0$ small enough such that $x_\delta(s_1) = 0$. \square

According to the lemma 3.1 and the lemma 3.2, we know that there exists $\delta_0 > 0$ small enough such that $x_{\delta_0}(s_1) = 0$ and $s_1(\delta_0) < \infty$. Since solutions of (2.10) depend smoothly on the initial value, we define δ^* as the following

Definition 3.1.

$$(3.29) \quad \delta^* = \sup\{\delta > 0 : x_\delta(s_1) = 0 \text{ and } s_1 = s_1(\delta) < \infty\}.$$

Lemma 3.3.

$$(3.30) \quad \sup_{\delta_0 < \delta < \delta^*} r_\delta(s_1) < +\infty.$$

Proof. From the lemma 3.2, we know $r_\delta(s_1)$ is bounded if $\delta > 0$ is small enough. If $r_{\delta_m}(s_1) \rightarrow +\infty$ for some sequence $\delta_m \rightarrow \delta^*$, we will prove that it is impossible. In fact, according to the mean value theorem, there exists an s_3 such that $0 < s_3 < s_1$ and $x'_{\delta_m}(s_3) = 0$ because of $x_{\delta_m}(s_1) = 0$. We should remark that s_1 and s_3 depend on δ_m . Furthermore, $x'_{\delta_m}(s) < 0$ for $s_3 < s < s_1$ and we have

$$\frac{n-1}{r_{\delta_m}} - r_{\delta_m} < 0,$$

for $s_3 < s < s_1$. Otherwise, there exists an s_4 such that $r_{\delta_m}(s_4) = \sqrt{n-1}$ with $s_4 > s_3$. Thus, $x'_{\delta_m}(s) < 0$ and $\frac{n-1}{r_{\delta_m}} - r_{\delta_m} < 0$ for $s_4 < s < s_1$. From the equations (2.10), we have

$$(3.31) \quad \kappa = -\frac{x''_{\delta_m}}{r'_{\delta_m}} = x_{\delta_m} r'_{\delta_m} + \left(\frac{n-1}{r_{\delta_m}} - r_{\delta_m}\right) x'_{\delta_m} + \lambda > \lambda.$$

By integrating (3.31) from s_4 to s_1 , we get

$$(3.32) \quad \begin{aligned} \frac{\pi}{2} &\geq \int_{s_4}^{s_1} \kappa ds \geq \lambda(s_1 - s_4) \\ &\geq \lambda(r_{\delta_m}(s_1) - r_{\delta_m}(s_4)) \\ &= \lambda(r_{\delta_m}(s_1) - \sqrt{n-1}). \end{aligned}$$

Hence, we have

$$(3.33) \quad r_{\delta_m}(s_1) \leq \frac{\pi}{2\lambda} + \sqrt{n-1}.$$

This is impossible since $r_{\delta_m}(s_1) \rightarrow +\infty$. Hence,

$$\frac{n-1}{r_{\delta_m}} - r_{\delta_m} < 0$$

for $s_3 < s < s_1$. For $s_3 < s < s_1$, we have

$$(3.34) \quad \kappa = -\frac{x''_{\delta_m}}{r'_{\delta_m}} = x_{\delta_m} r'_{\delta_m} + \left(\frac{n-1}{r_{\delta_m}} - r_{\delta_m}\right) x'_{\delta_m} + \lambda > \lambda.$$

Integrating (3.34) from $s = s_3$ to s_1 , we get

$$(3.35) \quad \frac{\pi}{2} \geq \int_{s_3}^{s_1} \kappa ds \geq \lambda(s_1 - s_3) \geq \lambda(r_{\delta_m}(s_1) - r_{\delta_m}(s_3)).$$

Hence,

$$(3.36) \quad r_{\delta_m}(s_1) \leq \frac{\pi}{2\lambda} + r_{\delta_m}(s_3).$$

We have

$$(3.37) \quad r_{\delta_m}(s_3) \rightarrow +\infty$$

because of $r_{\delta_m}(s_1) \rightarrow +\infty$.

Since $x'_{\delta_m}(s_3) = 0$ and $x_{\delta_m}(s_1) = 0$ hold, from $r_{\delta_m}(s_3) \rightarrow +\infty$, for some δ_m , which is very near δ^* , we know that there exists an s_5 with $s_3 < s_5 < s_1$ such that

$$(3.38) \quad x'_{\delta_m}(s_5) = -\sin\left(\frac{1}{\sqrt{r_{\delta_m}(s_3)}}\right).$$

If we integrate (3.34) from $s = s_3$ to s_5 , we obtain

$$(3.39) \quad \frac{1}{\sqrt{r_{\delta_m}(s_3)}} = \int_{s_3}^{s_5} \kappa ds \geq \lambda(s_5 - s_3).$$

Since $r_{\delta_m}(s_3) \rightarrow +\infty$ holds,

$$(3.40) \quad |x'(s_5)| = \sin\frac{1}{\sqrt{r_{\delta_m}(s_3)}} > \frac{1}{2\sqrt{r_{\delta_m}(s_3)}} > \frac{1}{2\sqrt{r_{\delta_m}(s_5)}}$$

yields

$$(3.41) \quad \begin{aligned} \left(\frac{n-1}{r_{\delta_m}(s)} - r_{\delta_m}(s)\right)x'_{\delta_m}(s) &\geq \left(\frac{n-1}{r_{\delta_m}(s_5)} - r_{\delta_m}(s_5)\right)x'_{\delta_m}(s_5) \\ &> \frac{n-1}{r_{\delta_m}(s_5)}x'_{\delta_m}(s_5) + \frac{1}{2}\sqrt{r_{\delta_m}(s_5)} \\ &> \frac{1}{4}\sqrt{r_{\delta_m}(s_5)}. \end{aligned}$$

for $s_5 \leq s \leq s_1$ since $r_{\delta_m}(s_5) \rightarrow +\infty$.

From the equations (2.10) and (3.41), we have

$$(3.42) \quad \kappa = -\frac{x''_{\delta_m}}{r'_{\delta_m}} = x_{\delta_m}r'_{\delta_m} + \left(\frac{n-1}{r_{\delta_m}} - r_{\delta_m}\right)x'_{\delta_m} + \lambda > \frac{1}{4}\sqrt{r_{\delta_m}(s_5)}$$

as $r_{\delta_m} \rightarrow \infty$. Integrating (3.42) from $s = s_5$ to s_1 , we have

$$(3.43) \quad \frac{\pi}{2} > \int_{s_5}^{s_1} \kappa ds > \frac{1}{4}\sqrt{r_{\delta_m}(s_5)}(s_1 - s_5).$$

Thus, one obtains from (3.39) and (3.43)

$$(3.44) \quad \begin{aligned} \max(x_{\delta_m}) &= x_{\delta_m}(s_3) = x_{\delta_m}(s_3) - x_{\delta_m}(s_1) \\ &\leq s_1 - s_5 + s_5 - s_3 \\ &\leq \frac{2\pi}{\sqrt{r_{\delta_m}(s_5)}} + \frac{1}{\lambda\sqrt{r_{\delta_m}(s_3)}} \\ &\leq \left(2\pi + \frac{1}{\lambda}\right) \frac{1}{\sqrt{r_{\delta_m}(s_3)}}. \end{aligned}$$

We conclude from (3.37) and (3.44) that $\max(x_{\delta_m}) = x_{\delta_m}(s_3) \rightarrow 0$ if $r_{\delta_m}(s_1) \rightarrow +\infty$. Hence, $\gamma_{\delta_m} = (x_{\delta_m}, r_{\delta_m})$ gets close to the r -axis, its tangents also must converge to the r -axis. It is impossible since $\gamma_{\delta_m} = (x_{\delta_m}, r_{\delta_m})$ converges to γ_{δ^*} which is not r -axis. This finishes our proof. \square

4. AN UPPER BOUND OF $x_\delta(s)$

Lemma 4.1.

$$(4.1) \quad \sup_{\delta_0 < \delta < \delta^*} \sup_{0 < s < s_1} x_\delta(s) \leq \frac{\pi}{2\lambda}.$$

Proof. For $\delta > 0$, letting $x'(s_3) = 0$ with $0 < s_3 < s_1$, we have

$$(4.2) \quad x'(s) > 0 \quad \text{for } 0 < s < s_3,$$

$$(4.3) \quad x'(s) < 0 \quad \text{for } s_3 < s < s_1.$$

If $r(s_3) \leq \sqrt{n-1}$, we see from $r'(s) > 0$ that $r(s) < \sqrt{n-1}$ for $0 < s < s_3$. Thus, it follows from (2.10) and (4.2) that

$$(4.4) \quad \kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > \lambda.$$

for $0 < s < s_3$.

By integrating (4.4) from $s = 0$ to s_3 , we have

$$(4.5) \quad \frac{\pi}{2} = \int_0^{s_3} \kappa ds \geq s_3 \lambda \geq \lambda x(s_3) = \lambda \sup_{0 < s < s_1} x_\delta(s).$$

$$(4.6) \quad \sup_{0 < s < s_1} x_\delta(s) = x(s_3) \leq \frac{\pi}{2\lambda}.$$

If $r(s_3) > \sqrt{n-1}$ for some $\delta > 0$, we have from $r'(s) > 0$ that $r(s) > \sqrt{n-1}$ for $s_3 < s < s_1$. Then it follows from (2.10) and (4.3) that

$$(4.7) \quad \kappa = -\frac{x''}{r'} = xr' + \left(\frac{n-1}{r} - r\right)x' + \lambda > \lambda$$

for $s_3 < s < s_1$.

By integrating (4.7) from $s = s_3$ to s_1 , we have, from the mean value theorem,

$$(4.8) \quad \frac{\pi}{2} \geq \int_{s_3}^{s_1} \kappa ds \geq \lambda(s_1 - s_3) \geq \lambda x(s_3) = \lambda \sup_{0 < s < s_1} x_\delta(s).$$

From (4.6) and (4.8), we get, for any $\delta > 0$,

$$\sup_{0 < s < s_1} x_\delta(s) = x(s_3) \leq \frac{\pi}{2\lambda},$$

that is,

$$\sup_{\delta_0 < \delta < \delta^*} \sup_{0 < s < s_1} x_\delta(s) \leq \frac{\pi}{2\lambda}.$$

This completes the proof of the lemma. \square

5. PROOF OF THE THEOREM

Proof. From the above lemmas, we have found that the profile curve $\gamma_\delta(s) = (x_\delta, r_\delta)$ ($0 \leq s \leq s_1$) stay away from the x -axis, and remain bounded as $\delta \rightarrow \delta^*$. Therefore, the limiting profile curve $\gamma^*(s) = \gamma_{\delta^*}(s)$ ($0 \leq s \leq s_1$) begins and ends on the r -axis, i.e., from $(0, \delta^*)$ to $(0, r^*)$, where $r^* = r_{\delta^*}(s_1)$.

We now claim that the profile curve has the horizontal tangent, that is, $x'_{\delta^*}(s_1) = -1$. From the definition of δ^* , we obtain that $x'_{\delta^*}(s_1) \leq 1$. If $x'_{\delta^*}(s_1) < 1$, one can choose δ such that $\delta > \delta^*$ and near δ^* . Then one can still obtain a profile curve γ_δ , which, in the first quadrant, is a graph $x = f_\delta(r)$, and which hits the r -axis in finite time. This contradicts the definition of δ^* . Hence, the profile curve γ^* has the horizontal tangent, that is, $x'_{\delta^*}(s_1) = -1$. We can get that the profile curve γ obtained by reflecting $\gamma_{\delta^*}([0, s_1])$ in the r -axis is a simple and closed curve in the upper half plane. This finishes our proof of the theorem 1.1. □

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